Slew Maneuver of Flexible Space Structures Using Time Finite Element Analysis

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Introduction

SINCE the birth of the time finite element method, many improvements have been suggested on its application in solving second-order initial value problems. Recently, Kim and Cho[1] developed a penalized weighted residual method that conforms well to the parallel computing scheme. Park and Kapania[2] studied several orthogonal polynomials to find the good basis function in the time finite element method. Suk and Kim[3] proposed a method to analyze the dynamic systems with specified initial and final conditions.

In this Note, the slew maneuver of a flexible space structure is analyzed using the time finite element method (FEM). The slew dynamics of a flexible space structure is governed by coupled equations in slew and flexible modes. Using time discretization, the slew mode equation can be converted to an algebraic matrix equation that incorporates the effect of structural flexibility and the control input torque. Also, the structural mode is reduced to a fourth-order partial differential equation that is a function of the solution of the slew mode. The novel feature of this study is that the original coupled system is split into two coupled sets of equations in which one set corresponding to the slew mode can be solved explicitly. Because the slew and structural modes of the flexible structure satisfy the characteristics of a self-adjoint system, it is also possible to reduce the order of the model by solving a generalized eigenvalue problem. To demonstrate and verify the proposed formulation, slew dynamics by open-loop control input is simulated and is compared with conventional FEM techniques.

System Equations of Motion

Consider the planar rotational/vibrational dynamics of a flexible structure consisting of a rigid hub with four cantilevered flexible appendages. The hybrid system of ordinary and partial differential equations governing the dynamics of this system is[4]

$$\rho\ddot{w}(x,t) + \rho x\ddot{\theta} + \frac{\partial^2}{\partial x^2}\left\{EI\frac{\partial^2 w(x,t)}{\partial x^2}\right\} = 0 \tag{1}$$

$$J_h\ddot{\theta} + 4\int_r^l \rho x[\ddot{w}(x,t) + x\ddot{\theta}]\,\mathrm{d}x = u_r(t) \tag{2}$$

where x is a spatial variable measured from the outer radius of the hub along the undeformed appendage axis, $\theta(t)$ is the slew angle, and $w(x,t)$ is the transverse deflection of the appendage measured from the x axis. Also, r is the radius of the hub, J_h is the moment of inertia of the hub, L is the length of the appendage, $l = r + L$, and $u_r(t)$ is the external torque generated by the reaction wheel mounted on the central hub. After multiplying Eq. (1) by δw and integrating the resulting equation by parts with initial and final conditions (rest-to-rest maneuver), we have

$$-\int_T \rho\dot{w}\delta\dot{w}\,\mathrm{d}t - \int_T \rho x\dot{\theta}\delta\dot{w}\,\mathrm{d}t + \int_T EIw^{(4)}\delta w\,\mathrm{d}t = 0 \tag{3}$$

Let us apply the time FEM to Eq. (3). The rigid-body and flexible coordinates are discretized using the shape functions $\phi_i(t)$ and $\psi_i(t)$ such that

$$w(x,t) = \sum_j \phi_j(t)w_j(x), \qquad \delta w(x,t) = \sum_j \phi_j(t)\delta w_j(x)$$

$$\theta(t) = \sum_i \psi_i(t)\theta_i, \qquad \delta\theta(t) = \sum_i \psi_i(t)\delta\theta_i$$

In this study, the quadratic functions are adopted as shape functions for simplicity. The following matrix ordinary differential equation for the structural mode is obtained from Eq. (3):

$$-\boldsymbol{M}\boldsymbol{w}(x) - \boldsymbol{M}\boldsymbol{\theta}x + \boldsymbol{K}\boldsymbol{w}^{(4)}(x) = 0 \tag{4}$$

where

$$M_{ij} = \int_T \rho\dot{\phi}_i(t)\dot{\phi}_j(t)\,\mathrm{d}t, \qquad K_{ij} = \int_T EI\phi_i(t)\phi_j(t)\,\mathrm{d}t$$

The matrix equation for the slew mode can be obtained by a similar approach as follows:

$$-\frac{J_{\text{tot}}}{\rho}\boldsymbol{M}\boldsymbol{\theta} - 4\boldsymbol{M}\int_r^l x\boldsymbol{w}(x)\,\mathrm{d}x = \boldsymbol{F}_r \tag{5}$$

where $J_{\text{tot}} = J_h + 4\rho(l^3 - r^3)/3$ and the ith element of the vector $\boldsymbol{F}_r$ is $(1/\rho)\int_T u_r(t)\phi_i(t)\,\mathrm{d}t$.

Model Reduction

Spatial distribution vector $\boldsymbol{w}(\boldsymbol{x})$ in Eq. (4) may have lots of time elements, and this may act as a computational burden. To reduce the computation effort without significantly compromising the accuracy, model reduction should be performed. To do this, spatial distribution vector $\boldsymbol{w}(\boldsymbol{x})$ should first be transformed into time-based modal coordinates. Consider the following generalized eigenvalue problem:

$$MP = KP\Lambda \tag{6}$$

where Λ is the diagonal matrix composed of the generalized eigenvalues and P is the modal matrix consisting of the generalized eigenvectors. Note that the column vectors of P form a basis that describes the time response of the flexible appendage. Let us introduce the modal coordinate transformation

$$\boldsymbol{w}(x) = P_\eta\boldsymbol{\eta}(x), \qquad \boldsymbol{\theta} = P_\xi\boldsymbol{\xi} \tag{7}$$

where P_η and P_ξ are the modal matrices that retain a specified number of eigenvectors, starting from the lowest mode, and the following normalization equations are used:

$$P_\eta^T KP_\eta = I_\eta, \qquad P_\xi^T KP_\xi = I_\xi$$

$$P^T MP_\eta = \Lambda_\eta, \qquad P_\xi^T MP_\xi = \Lambda_\xi$$

Using modal coordinate transformation, the reduced-order model is obtained as follows:

$$\boldsymbol{\eta}^{(4)}(x) = \Lambda_\eta\boldsymbol{\eta}(x) + P_\eta^T MP_\xi\boldsymbol{\xi}x \tag{8}$$

$$\frac{J_{\text{tot}}}{\rho}\Lambda_\xi\boldsymbol{\xi} + 4P_\xi^T MP_\eta\left\{\int_r^l x\boldsymbol{\eta}(x)\,\mathrm{d}x\right\} = -P_\xi^T F_r \tag{9}$$

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Spatial Propagation Equation for the Appendage

Equation (8) can be transformed into the first-order system

$$y' = Ay + Bx \tag{10}$$

where

$$A = \begin{bmatrix} 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & \boldsymbol{I} \\ \Lambda_\eta & 0 & 0 & 0 \end{bmatrix}, \qquad B(\boldsymbol{\xi}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ P_\eta^T M P_\xi \boldsymbol{\xi} \end{bmatrix}$$

$$y = \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\eta}' \\ \boldsymbol{\eta}'' \\ \boldsymbol{\eta}''' \end{bmatrix}$$

The response of the flexible appendage at an arbitrary point can be obtained as follows:

$$y(x) = e^{A(x-r)} y(r) + \int_r^x e^{A(x-\zeta)} B\zeta \, \mathrm{d}\zeta, \qquad r \le x \le l \tag{11}$$

$$= e^{A(x-r)} y(r) + \boldsymbol{\Phi}\Sigma(x)\boldsymbol{\Phi}^{-1} B(\boldsymbol{\xi}) \tag{12}$$

where $\boldsymbol{\Phi}$ is the modal matrix of A, and if λ_i are the ith eigenvalues of A, then

$$\Sigma(x) = \mathrm{diag}\{\ldots, \sigma_i(x), \ldots\}$$

$$\sigma_i(x) = e^{\lambda_i(x-r)}\left\{\left(r/\lambda_i\right) + \left(1/\lambda_i^2\right)\right\} - \left\{\left(x/\lambda_i\right) + \left(1/\lambda_i^2\right)\right\}$$

Now the motion of the tip of the flexible appendage can be obtained as follows:

$$y(l) = e^{AL} y(r) + \boldsymbol{\Phi}\Sigma(l)\boldsymbol{\Phi}^{-1} B(\boldsymbol{\xi})$$

$$\equiv \Omega y(r) + \boldsymbol{\Psi} B(\boldsymbol{\xi}) \equiv \Omega y(r) + \mu(\boldsymbol{\xi}) \tag{13}$$

Let us define new partitioned state vectors

$$y_1(x) = \begin{bmatrix} \boldsymbol{\eta}(x) \\ \boldsymbol{\eta}'(x) \end{bmatrix}, \qquad y_2 = \begin{bmatrix} \boldsymbol{\eta}''(x) \\ \boldsymbol{\eta}'''(x) \end{bmatrix}$$

Using the preceding state vectors, Eq. (13) can be rewritten as

$$\begin{bmatrix} y_1(l) \\ y_2(l) \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} y_1(r) \\ y_2(r) \end{bmatrix} + \begin{bmatrix} \mu_1(\boldsymbol{\xi}) \\ \mu_2(\boldsymbol{\xi}) \end{bmatrix} \tag{14}$$

With fixed-free boundary conditions of the appendage, i.e., fixed at $x = r$ and free at $x = l$, we have

$$y_2(r) = -\Omega_{22}^{-1}\mu_2(\boldsymbol{\xi}) \tag{15}$$

$$y_1(l) = \Omega_{12} y_2(r) + \mu_1(\boldsymbol{\xi}) \tag{16}$$

Note that, in Eqs. (15) and (16), all of the state vectors are represented as a function of $\boldsymbol{\xi}$. The information of vector $\boldsymbol{\xi}$ can be obtained by Eq. (9). Then all of the spatial distributions of the mechanical characteristics of the slewing appendages can be determined by Eq. (11). To obtain the solution of Eq. (9), the integration

$$\int_r^l x\boldsymbol{\eta}(x)\,\mathrm{d}x$$

should first be transformed into a function of ξ. The following equation can be obtained by using Eq. (11):

$$\int_r^l x y(x)\,\mathrm{d}x = \int_r^l e^{A(x-r)} x y(r)\,\mathrm{d}x + \int_r^l \boldsymbol{\Phi}\Sigma(x)\boldsymbol{\Phi}^{-1} B(\boldsymbol{\xi}) x\,\mathrm{d}x$$

$$\equiv \boldsymbol{\Phi}\Theta\boldsymbol{\Phi}^{-1} y(r) + \boldsymbol{\Phi}\Xi\boldsymbol{\Phi}^{-1} B(\boldsymbol{\xi}) \tag{17}$$

where Θ and Ξ are the diagonal matrices, respectively, whose diagonal terms are represented as follows:

$$\Theta_{ii} = e^{\lambda_i L}\left\{\frac{l}{\lambda_i} - \frac{1}{\lambda_i^2}\right\} - \left\{\frac{r}{\lambda_i} - \frac{1}{\lambda_i^2}\right\}$$

$$\Xi_{ii} = -\left(\frac{l^3 - r^3}{3\lambda_i} + \frac{l^2 - r^2}{2\lambda_i^2}\right) + \left\{\frac{r}{\lambda_i} + \frac{1}{\lambda_i^2}\right\} \times \left\{e^{\lambda_i L}\left(\frac{l}{\lambda_i} - \frac{1}{\lambda_i^2}\right) - \left(\frac{r}{\lambda_i} - \frac{1}{\lambda_i^2}\right)\right\}$$

Therefore, by substituting Eq. (15) into Eq. (17),

$$\int_r^l x\boldsymbol{\eta}(x)\,\mathrm{d}x$$

can be obtained as

$$\int_r^l x\boldsymbol{\eta}(x)\,\mathrm{d}x = \left\{\Pi_{11,44} - \Gamma_{11,34}\Omega_{22}^{-1}\Psi_{34,44}\right\} P_\eta^T M P_\xi \boldsymbol{\xi} \tag{18}$$

where $\Pi = \Phi\Xi\Phi^{-1}$; $\Gamma = \Phi\Theta\Phi^{-1}$; and matrices $\Pi_{ij,kl}$, $\Gamma_{ij,kl}$, and $\Psi_{ij,kl}$ are submatrices consisting of ij rows and kl columns of Π, Γ, and Ψ, respectively, when the individual matrix is divided into 4×4 square submatrices.

Slew Maneuver for the Open-Loop Control Input

Using Eqs. (9) and (18) in Eq. (7), we obtain the following equation:

$$\theta = -P_\xi\left[\left(J_{\mathrm{tot}}/\rho\right)\Lambda_\xi + 4P_\xi^T M P_\eta\left(\Pi_{11,44} - \Gamma_{11,34}\Omega_{22}^{-1}\Psi_{34,44}\right) P_\eta^T M P_\xi\right]^{-1} P_\xi^T \boldsymbol{F}_r \tag{19}$$

For the given control input $u_r(t)$, the time response of the slew mode can be solved explicitly from Eq. (19), and the motion of appendage can be obtained from Eqs. (10) and (7) with initial conditions. Now the total response of a mass on the appendage can be evaluated by

$$z(x, t) = w(x, t) + (x + r)\theta(t) \tag{20}$$

Numerical Results

A numerical example is shown to validate the proposed method, and the results are compared with those obtained by the conventional spatial domain FEM. For this example, the number of time finite elements is 100, and dynamic analyses are performed using the one-fifth reduced time model to show the effect of the substantial

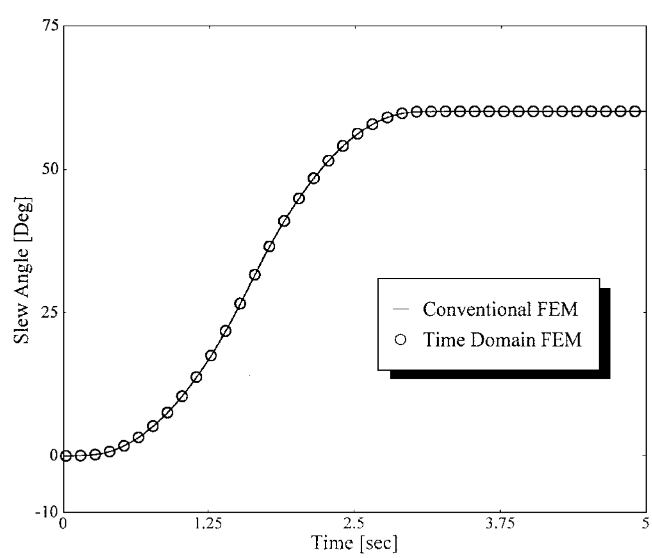

Fig. 1 Slew angle.

model reduction in time. The reference input profile $u_{\text{ref}}(t)$ for the slew maneuver is represented by a Fourier sine series expansion as[5]

$$u_{\text{ref}}(t) = \sum_j a_j \sin \frac{2\pi j t}{t_f}$$

The coefficients of the trigonometric series are adopted as design variables to determine the input shape by the optimization procedure. The input coefficients are obtained such that they minimize the vibration energy of the appendage during the maneuver while satisfying the constraint equations to guarantee the rest-to-rest maneuver. Optimized input shaping parameters are computed as $a_1 = 1.1180$, $a_2 = -0.0857$, $a_3 = 0.2714$, $a_4 = -0.1437$, $a_5 = 0.0731$, $a_6 = -0.1680$, and $a_7 = 0.0006$. Figures 1 and 2 show the comparative results for the slew angle and the time trajectories of the appendage, respectively. Note that the rest-to-rest maneuver guarantees zero initial displacement for both slew and flexible modes. It is also seen from Fig. 2 that the dynamic motion of the system can be described very well with only 40 modes among 200 time modes.

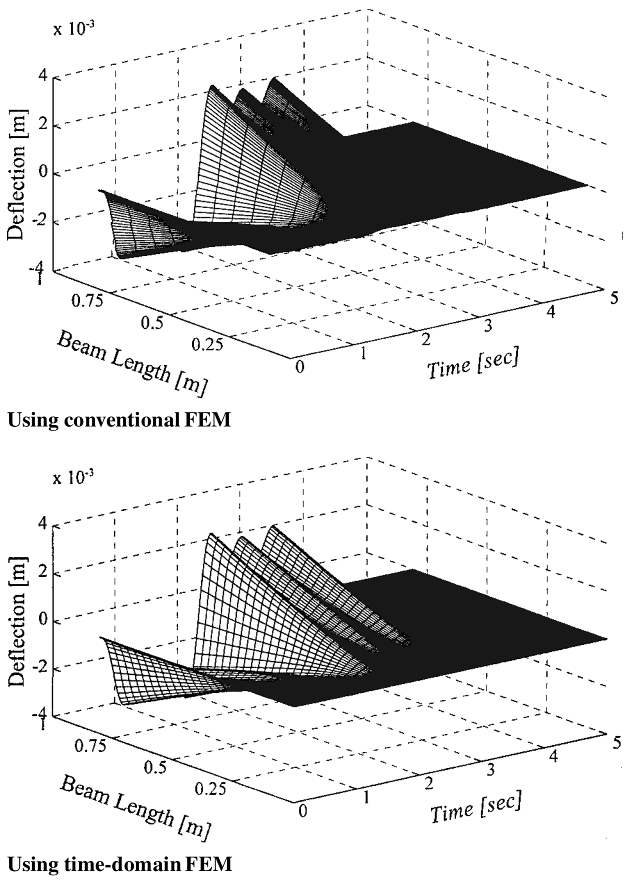

Fig. 2 Three-dimensional plot for open-loop control.

Conclusion

The slew maneuver of the flexible space structure is analyzed using the time FEM. Using time discretization, the slew mode equation is converted to an algebraic matrix equation that can be solved explicitly. Because of the coupling effect of the slew mode with flexible mode, the time trajectories of the structural mode are represented as a function of the solution of the slew mode. The procedure for efficient model reduction is proposed by using the self-adjoint properties of the time-based modal coordinates. The number of time modes can be selected to simulate the motion of the system in consideration of the accuracy and computational efficiency. Open-loop responses for the slew maneuver are computed to verify the proposed method and are compared with conventional FEM techniques.

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